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Henry Lewis Rietz 1875-1943

Evaluation of a Limit by a Sequence of Triangles

Gauss and the Early Development of Algebraic Numbers

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Henry Lewis Rietz
1875-1943

This is a brief statement in memory and in honor of the life and labors of Professor Rietz. No effort is made to give a bibliography or account in detail of his many merits as a man and mathematician. No one paper, nor even one volume, could contain all that should be written of this versatile pioneer among the great American mathematicians of our day.

For the record we note that the Ph.D in mathematics was conferred at Cornell University (1902), where he was graduate assistant in mathematics. He was Professor of Mathematics and Astronomy at Butler College (1903); Instructor and Professor at the University of Illinois (1903-1918); and Head of the Department of Mathematics at the University of Iowa (1918-1943). Besides holding these professional positions with many distinguished honors he was Actuary and Consultant to organizations of national importance among which are: The University of Illinois Experiment Station; Illinois Pension Commission; Chicago Pension Commission; Teacher's Insurance and Annuity Association; and the President's National Committee on Economic Security. His published papers appear in many mathematical and statistical journals around the world. More than 150 papers in the fields of pure mathematics and applied mathematics exhibit the remarkable versatility and high quality of his productive career. To this list we add eleven books which place among the most widely used mathematics text books and reference books. At the University of Iowa he directed 14 Doctor's Dissertations as follows: Reilly, John Franklin, (1921); Weida, Frank M. (1923); Smith, Clarence DeWitt (1928); Meyer, Herbert A. (1929); Craig, Allen Thornton (1931); Wilks, Samuel Stanly (1931); Fischer, Carl H. (1932); Harper, Floyd S. (1935); Ollivier, Arthur (1935); Kowler, Lloyd A. (1937); Olshen, Abraham C. (1937); Berg, William D. (1941); Satterthwaite, Franklin

(1941); Garfin, Lewis (1942). The last two were completed with the assistance of Professor Craig. He was a member of all of our national mathematical organizations and held high positions in each. He served on the editorial board of each of the official journals. His fraternal connections included three honorary scientific fraternities and the Alpha Tau Omega social fraternity.

One can hardly understand how a man with so many professional achievements could find time for active participation in organizations for display of the social and moral virtues. Yet the record of his life includes the following partial list of social and religious activities: Chairman Finance Committee of his Country Club for more than 15 years and President two years; Charter member of Iowa City Building and Loan Association and President of the association for more than 20 years; Board of Directors First Capitol State Bank of Iowa City; President of Kiwanis Club; Member of official board and chairman of finance committee of the Methodist church of Iowa City for more than 20 years.

Perhaps a few words of personal testimony may add to the facts of his life work some conception of the way Professor Rietz liked to live among his friends. During the years 1924-1928 he directed the program of my graduate studies. I was privileged to know something of his personal qualities and to see how he welcomed the social companionship of his students. In all professional matters he was exacting to the last minute detail. No question of doubt regarding the truth of a proposition or the rigor attending a demonstration was ever overlooked. He was a master of language. In reading his papers one must be impressed by the elegance, beauty, and clarity with which he wrote. Always jealous for the progress of his students he would exhibit almost childish glee when one achieved a new proposition. He often came to my desk to give me theater tickets for my family or to invite me to his golf club. I knew him at leisure, at golf, at home, at the Union Club lounge, and at his office during week ends. He delighted in discussion of situations which I met in the U. S. Army and things that came his way in Washington. Amusing incidents and jokes always had a special appeal. Yes, I knew Henry Lewis Rietz as a master in his field, and as a lovable character whose moral and religious virtues stood the test of arduous living, long illness, and the approach of inevitable death.

This account would not be complete without an expression of gratitude for the kind interest he showed in NATIONAL MATHEMATICS MAGAZINE. The magazine was placed on his list for the library and he always displayed sympathetic interest in this journal. He believed in the maintenance of mathematical publications for development of mathematics at all levels everywhere. The Institute of Mathematical Statistics and publication of the Annals of Mathematical Statistics were due in a large measure to his efforts. NATIONAL MATHEMATICS MAGAZINE along with every journal for promotion of scientific knowledge has lost a zealous friend. We cannot record the value of such a life. We can only cherish his memory as we feel the impact of a great loss.

Mississippi State College. C. D. SMITH.

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By DR. GLENN R. JAMES and R. C. JAMES

RECALLS ARITHMETIC AND SUBSEQUENT SUBJECTS IN CLEAR, CONCISE LANGUAGE

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Evaluation of a Limit by a Sequence of Triangles

By C. E. SPRINGER
University of Oklahoma

Consider the polynomial

$$\prod_{i=0}^{n-1}(x+i)\equiv\sum_{j=0}^{n-1}\sigma_j(n-1)x^{n-j}\ , \tag{1}$$

where $\sigma_j(n-1)$ denotes the sum of the products of the integers from 1 to $n-1$, taken j at a time. (The symbol $\sigma_0(n-1)$, is defined to be unity.) If x is put equal to 1, there results

$$\prod_{i=0}^{n-1}(1+i)=n!=\sum_{j=0}^{n-1}\sigma_j(n-1). \tag{2}$$

Let a_n, b_n, c_n denote the respective partial sums of the terms in the last member of (2) for which $n-j\equiv 0, 1, 2,$ (modulo 3). That is,

$$a_n,\ b_n,\ c_n=\sum_{j=0}^{n-1}\sigma_j(n-1),\quad n-j\equiv 0,\ 1,\ 2 \text{ (modulo 3)}. \tag{3}$$

It is the purpose of this note to evaluate the limits of the ratios

$$\frac{a_n,\ b_n,\ c_n}{n!} \tag{4}$$

as n becomes infinite through positive integers.

This will be effected by a study of the areas of a sequence of triangles $A_kB_kC_k$ $(k=0, 1, 2, \cdots)$ defined as follows: The first triangle $A_0B_0C_0$ of the sequence is arbitrary. On the side A_0B_0 of the first triangle take A_1 so that it divides A_0B_0 in the ratio 1:0. B_1 and C_1 are taken similarly on sides B_0C_0 and C_0A_0. Take A_2 so that it divides side A_1B_1 of triangle $A_1B_1C_1$ in the ratio 1:1. Similarly, for B_2 and C_2. In general, take A_{n+1} so that it divides side A_nB_n of triangle $A_nB_nC_n$ in the ratio $1:n$. Points B_{n+1} and C_{n+1} are taken similarly on B_nC_n and C_nA_n. The limit of the area of the nth triangle $A_nB_nC_n$ as n becomes infinite through integers will now be considered.*

*J. Rosenbaum considered a similar question, *American Mathematical Monthly*, 1937, p. 386.

If $A_0B_0C_0$ is chosen as triangle of reference for areal coordinates with (1, 0, 0), (0, 1, 0), (0, 0, 1) as coordinates of A_0, B_0, C_0, the coordinates of the vertices $A_k(k=1,\cdots,5,\cdots)$ are given by

$$(5)\qquad A_1(0,1,0),\ A_2\left(\frac{0,1,1}{2!}\right),\ A_3\left(\frac{1,2,3}{3!}\right),$$

$$A_4\left(\frac{6,7,11}{4!}\right),\ A_5\left(\frac{35,34,51}{5!}\right),\cdots$$

If the coordinates of the vertex A_n are denoted by

$$\left(\frac{a_n,\ b_n,\ c_n}{n!}\right),$$

the coordinates of B_n and C_n can be written by cyclic permutation as

$$\left(\frac{c_n,\ a_n,\ b_n}{n!}\right)\quad\text{and}\quad\left(\frac{b_n,\ c_n,\ a_n}{n!}\right).$$

The difference equations which generate (5), namely,

$$(6)\qquad a_{n+1}=na_n+c_n,\quad b_{n+1}=nb_n+a_n,\quad c_{n+1}=nc_n+b_n,$$

are readily found to obtain. Also, because of the nature of areal coordinates, the relation

$$(7)\qquad a_n+b_n+c_n=n!$$

holds for every n.

It will now be proved by induction that the numbers a_n, b_n, c_n in the coordinates of A_n are the numbers given in (3). This is readily verified for small values of n. For example, if $n=4$,

$$\prod_{i=0}^{3}(x+i)=\sum_{j=0}^{3}\sigma_j(3)x^{4-j}=x^4+6x^3+11x^2+6x,$$

and we have $a_4=6$, $b_4=1+6$, $c_4=11$, which are the numbers appearing in the coordinates for A_4 as given by (5). If equation (1) is multiplied by $x+n$, there results

$$(8)\qquad \prod_{i=0}^{n}(x+i)\equiv\sum_{j=0}^{n}\left[\sigma_j(n-1)+n\sigma_{j-1}(n-1)\right]x^{n+1-j}=\sum_{j=0}^{n}\sigma_j(n)x^{n+1-j}.$$

By the difference relations (6) it is seen that the numbers a_{n+1}, b_{n+1}, c_{n+1} are respectively the sums of the coefficients of x^{n+1-j} in (8) for which $n+1-j\equiv 0, 1, 2$ (modulo 3).

The three coordinates of the centroid of $A_nB_nC_n$ are each equal to $(a_n+b_n+c_n)/3n!$, which is 1/3 by means of (7). Therefore, all triangles of the sequence have the fixed point (1, 1, 1)/3 for centroid.

Attention is now directed to the ratio R_n of the area of triangle $A_nB_nC_n$ to that of the reference triangle $A_0B_0C_0$. This is readily shown to be given by

$$R_n = 1 - 3\,\frac{b_nc_n + c_na_n + a_nb_n}{(n!)^2}. \tag{9}$$

On replacing n by $n+1$, and on using the difference relations (6), it is found that

$$\frac{R_{n+1}}{R_n} = 1 - 3\,\frac{n}{(n+1)^2}. \tag{10}$$

Hence, the value of R_n is given by

$$R_n = \prod_{k=1}^{n} \left(1 - 3\frac{k-1}{k^2} \right). \tag{11}$$

The limit of R_n as n becomes infinite is zero because the series

$$\sum_{k=1}^{\infty} \frac{k-1}{k^2} \tag{12}$$

is divergent. Therefore, since the area of $A_nB_nC_n$ becomes zero as n becomes infinite, and since the centroid of all triangles of the sequence is a fixed point, the vertices A_n, B_n, C_n approach coincidence at the point (1, 1, 1)/3 as n becomes infinite.

Hence, the ratio of the sum of all the coefficients in polynomial (1) for which $n - j \equiv r$ (modulo 3) to $n!$ has the limit 1/3 as n becomes infinite, where $r = 0$, or 1, or 2.

Humanism and History of Mathematics

Edited by
G. WALDO DUNNINGTON and A. W. RICHESON

Gauss and the Early Development of Algebraic Numbers

By E. T. BELL
California Institute of Technology

1. *From arithmetic to abstract algebra.* An unexpected turn in twentieth-century mathematics was the abrupt change in the motivation and objectives of algebra. The change became evident by 1925 at the latest, and in about ten years made some of the algebra of the nineteenth and early twentieth centuries seem rococo and strangely antiquated to algebraists of the younger generation.

The transition from individually developed theories, overloaded with masses of intricate theorems—often the seemingly fortuitous outcome of elaborate calculations carried through with consummate manipulative skill—to the deliberate search for unifying abstract principles was sudden. It did not occur without preparation, of course; but the passage from calculation to preoccupation with fundamental concepts was accomplished within a decade.

Without the vast accumulations of special results, like the data in Darwin's notebooks which suggested the theory of evolution, the problems and methods of abstract algebra might never have emerged. However that may be, interest in the algoristic type of algebra declined rapidly after 1921. Theorems that had been obtained primarily by modes of calculation germane to the particular theory in which the theorems originated, were seen to be instances of underlying structures independent of the particular theory. To cite but one striking example, the Jordan-Hölder theorem and its refinements became more intuitive when analyzed structurally than they had seemed in their traditional settings.

Actually the change might have happened earlier than it did. Hilbert's basis theorem[1] of 1890, in both its statement and its proof, was plainly in an order of ideas different from that of the algebra of

its epoch. Yet earlier, in 1877, Dedekind clearly formulated the strategy of abstract algebra, in a statement[2] of the methodology which he had applied in developing his theory of ideals. Having noted that his initial attempt at a general theory of algebraic numbers could, conceivably, be successfully completed, but only (he believed) by overcoming serious obstacles in calculation, Dedekind continued as follows:

"But even if a theory [does not encounter great difficulties when developed algoristically], it seems to me that such a theory, based on calculation, still does not offer the highest degree of perfection. As in the modern theory of functions, it is preferable to seek to extract proofs, not from calculation, but immediately from characteristic fundamental concepts, and to construct the theory in such a manner that, on the contrary, it shall be in a position to predict the results of calculation."

That declaration of intention might well stand as the party manifesto of the modern abstract algebraists. Dedekind himself gave numerous and brilliant examples of his grand strategy. A typical simple specimen, which has passed unaltered into current usage, is his postulational definition of an ideal, and there are many others.

As one of the matters which Dedekind had in mind while distinguishing between mathematics and calculation will appear in the sequel, its nature may be indicated here. It was the theory of composition of forms (homogeneous polynomials) and its outgrowth in that chapter of the classical theory of algebraic number ideals which is devoted to factorable forms of degree n in n indeterminates. Of great historical interest on account of the famous mathematicians who observed some aspect of the whole, but who could not possible have given anything approaching an adequate discussion with the mathematics of their respective eras, the theory of composition still retains its mathematical interest in problems yet unsolved. Its particular significance here, however, is that it was partly responsible for Dedekind's abandonment of calculation in favor of the abstract approach. From Diophantus (c. 250 A. D.) to Brahmegupta (7th century) to Leonardo of Pisa (13th century), to Euler (18th century), to Lagrange (18th century), to Gauss (1777-1855), to Kummer (1810-1893), scattered instances of composition kept foreshadowing something beyond, till Dedekind (1831-1916), rejecting the generalization suggested by Gauss' development for binary quadratic forms and effected by Kummer for cyclotomic fields, abandoned calculation and extracted his proofs "immediately from characteristic fundamental concepts."

To prevent possible misconceptions, it may be observed that few algebraists imagine the current abstract phase to be the climax of algebra. Many things are now much clearer and simpler than they

seemed a generation ago, and much has been exposed that was unsuspected then. But the simplicity and clarity frequently refer to theories first worked out by the hard way of calculation from basic discoveries made almost empirically in the course of routine labor. The same kind of drudgery on the higher level of abstractions has been in part responsible for some of the recent accessions; and a single new observation at any time may start another cycle of calculation, discovery, abstraction, calculation. Like the rest of mathematics, algebra needs an occasional transfusion of fresh ideas to keep alive. Nor can algebra go on living on its past forever.

Anyone following the historical evolution of abstract algebra much be impressed by the frequency with which the development was influenced by concepts whose germs, at least, appeared first in the theory of algebraic number fields and their rings of integers. Intensive scrutiny of classical arithmetical ideas, such as integer, unit, prime, divisibility, equivalence relation, domain of integrity, ring, field, module, residue class, ideal, dual group, and many more, suggested abstractions relevant for algebra as a whole.

In the process of abstraction, precise axiomatic technique applied to such arithmetical concepts yielded new ideas, refinements of the concepts from which they may first have been abstracted, and these in turn enriched arithmetic with thitherto unsuspected subtleties. A case in point was the introduction of primary ideals in the theory of commutative rings, which revitalized and amplified a theory whose rudimentary form is as old as Euclid. Another was the outgrowths of Dedekind's uses of mapping and correspondences in a discrete domain, that of algebraic numbers.[3)]

Granted that the theory of algebraic numbers has been a suggestive lead in the development of abstract algebra, it may be of interest to see how much of subsequent progress was adumbrated in the work of Gauss, universally acknowledged as the initiator of the theory. To appreciate in some slight degree the novelty and the magnitude of Gauss' achievement in introducing complex (Gaussian) integers into arithmetic, it will be advantageous first to consider briefly the nature of the general problem of which the one that he solved is a very special case.

2. *The nature of Gauss' problem.* The theory of algebraic numbers is a natural but by no means immediate extension and generalization of rational arithmetic. The primary concern of rational arithmetic is the study of relations between the rational integers.

In any generalization of rational arithmetic, certain fundamentals must be agreed upon before the generalization can significantly be

called arithmetic. In particular, what phenomena of rational arithmetic are to persist in the generalization? The requirement that the generalization, when suitably specialized, shall yield at least part of the classical theory of numbers for rational integers, has been admitted in all generalized arithmetics thus far constructed.

The remaining demands, either explicitly or by tacit assumption, refer to the multiplicative department of arithmetic. The more important of these, in the historical development, appear to have been the distinction between integral and non-integral elements; prime integral elements; unit integral elements; or, concisely, integers, primes, units.

It has not always been immediately evident how the integers, primes, and units were to be selected or defined in some species of algebra for which an arithmetic was desired. The instance of quaternions may be recalled. Generally, the distinction between algebra and arithmetic has been roughly summarized in the dictum that division is only exceptionally impossible in algebra and only exceptionally possible in arithmetic. The quotient of two complex numbers, for example, is a complex number; the quotient of one algebraic integer by another is an algebraic integer only if the second is a divisor of the first. If the elements or generalized "numbers" concerned form a ring with respect to suitably defined operations of "addition" and "multiplication", the integers may be isolated by a system of postulates embodying those characteristics of rational integers which it is desired to preserve. If the addition and multiplication of integral elements do not satisfy all the postulates for the similarly named operations in the ring of rational integers, there is as yet no common agreement on the postulates for integers. Different authors have favored different definitions.

Incidentally, this illustrates the distinction in mathematics between naming a thing and defining it. In the early development of algebraic numbers, the integers were named, but either not defined or defined only inferentially. Explicit definition was not insisted upon until the twentieth century, and then not always. The desirability of a postulational definition of algebraic integers in the first great generalization of rational arithmetic was overlooked, possibly on account of the relative simplicity and familiarity of the material.

With the analysis of divisibility the necessity for precision became apparent. The distinction between primes and indecomposable or irreducible integers was first sharply recognized by Dirichlet,[4] about 1843, although probably Gauss had noticed it long before he officially introduced[5] (1831) his complex integers $a+bi$, a, b rational integers, into arithmetic. The distinction being of no significance for rational

arithmetic, it proved singularly elusive. Actually it slipped through the grasp of mathematicians as powerful as Lamé[55]* and Cauchy.[57]*

With primes satisfactorily defined, the next capital question was that of units. The definitions of primes and units are interdependent, and this may have concealed the more profitable approach at first. The "natural" definition that a unit is an integer which divides (arithmetically) every integer, may have been easy to guess. But this natural definition is comparatively sterile. Its more productive equivalent in terms of norms, with which Gauss as early as 1808 was acquainted in special cases, can scarcely have been obvious.

There would have been but little point in defining primes and units unless they were to provide a basis for unique factorization. Fortunately for Gauss, in applications of his complex integers to cubic and biquadratic residues,[6] his primes satisfied this demand: up to unit factors a Gaussian integer is uniquely the product of a finite number of powers of primes in the integral domain defined by $i^2+1=0$, and likewise for the domain defined by $\rho^2+\rho+1=0$. This relative unicity however did not wholly satisfy the exacting Gauss. To appease his arithmetical conscience, Gauss insisted on absolute uniqueness, and attained it for Gaussian integers by a proper classification of associates.[7]

In all of his work on algebraic numbers thus far recovered, Gauss relied on the existence of the Euclidean algorithm, for the greatest common divisor of two integers, to give him his unique factorization proofs. The Euclidean algorithm is a sufficient but not necessary condition for unique factorization. Whether Gauss was aware of this distinction seems not to be known.

Progress in the theory of algebraic numbers has gone so far since Gauss took the first bold steps into "a boundless new domain of arithmetic", that the foregoing remarks on integers and the rest may strike a beginner today as platitudes and truisms. Possibly they are; but if so, it should be an easy exercise to construct an acceptable arithmetic for a general skew ring, or to show that there is none.

3. *The central difficulty.* The anticipations of non-Euclidean geometry and elliptic functions by Gauss are as well established as they are well known. His actual contributions to the early development of algebraic numbers are readily accessible and equally well known. But, so far as seems to have been discovered, Gauss left no such illuminating record of his thoughts on algebraic numbers as he did for elliptic functions and non-Euclidean geometry. Nor, when Kummer[8] in 1845 restored the fundamental theorem of arithmetic

*These notes will appear in the next number.—EDITOR.

(finite and unique factorization into primes) to his cyclotomic fields by the invention of ideal numbers, did Gauss give any sign that he may have been blocked by the central difficulty—the breakdown of the fundamental theorem—in his own researches of a quarter-century earlier.

It would be of interest to know whether Gauss did encounter that apparently insurmountable obstacle—Kummer[9] seems at first (1844) to have despaired of surmounting it—and whether, if he did, he hoped it might be obviated. From what has survived of Gauss' early work on algebraic numbers, it is possible that he recognized the main difficulty but foresaw no possibility of circumventing it. Certainly any way that he may have tried—to judge by what he himself published and the posthumous fragments and correspondence whose publication he could not have foreseen—must have been quite different from Dedekind's ideals. And for reasons that will appear in connection with Galois imaginaries, it is unlikely that Gauss would have recognized Kummer's ideal numbers as legitimate arithmetical concepts, even if they had occurred to him.

Gauss might, conceivably, have started from the higher theory of congruences, which was to have formed the eighth section of the *Disquisitiones Arithmeticæ*. Dedekind[2] at one time believed the end attainable by this approach, but after much thought decided (erroneously) that this was not a feasible way. It was not until the twentieth century that a usable theory[59]* in this general direction was constructed. It is not likely that Gauss could have progressed so far, although he does throw out hints for overcoming some of the difficulties he leaves unclarified in his theory of congruences to a double modulus.[48]* Still less likely is it that he ever imagined anything like Hensel's theory of p-adic numbers, which provides a theoretical means for finding the discriminant of an algebraic field, but which is difficult to apply. It is scarcely enough in the theory of numbers to prove that a problem can be solved in a finite number of steps, if the time required for taking the necessary steps seriously encroaches on eternity.

That Gauss was aware of at least some of the pitfalls in algebraic number fields as early as 1816 appears—though somewhat cryptically—in a letter (to be quoted later) of that year. Indeed, as will be seen presently, Euler in 1770 had observed one of the apparent anomalies in the behavior of algebraic numbers, in a work with which Gauss was familiar almost from his boyhood. This alone would have sufficed as a signal to proceed with caution.

Some of Gauss' earliest work in arithmetic and algebra can now be re-read in the light of its much later significance. As examples of

*These notes will appear in the next number.—EDITOR.

unconscious foresight, the relevant matters are among the most interesting in the history of nineteenth-century mathematics. These will be noted next.

4. *Shapes of things to come.* Gauss (1777-1855) began the *Disquisitiones Arithmeticæ* early in 1795. It went to press in April, 1798, and was published in September, 1801. During the printing Gauss made several corrections and additions.

Although the work contains nothing on the theory of algebraic numbers in the current technical sense, it is rich in ideas and theorems that were to prove suggestive to the creators of the theory. The seventh section, on cyclotomy, concerned with the roots of binomial equations though it is, is no exception. The algebraic irrationalities appearing are not recognized as arithmetical beings worth or capable of sharing the austere purity of the rational integers. "The higher arithmetic" for Gauss in 1801 meant a theory of the rational integers and nothing more.[10] Rational fractions were tolerated on occasion as removable blemishes. If Gauss in his private thoughts speculated on the possibility of admitting other "numbers" to the select society of the rational integers, he permitted no hint of what he was thinking to escape into print.

A few items from the many in the *Disquisitiones* and other early works, that were to acquire a broadened meaning when algebraic numbers were recognized as legitimate material for arithmetic, may be recalled as specially significant. There is no question here of anticipations. Gauss never hinted that he had foreseen the potentially wider scope of his early work; nor has any mathematical critic, however partisan in other judgments, asserted that he did.

First in order and second to none in importance, is the proof (Art. 13) of the fundamental theorem of arithmetic—finite and unique decomposition into primes. Until Gauss proved this theorem, it either had been overlooked or taken for granted as obvious. Gauss gave a careful proof, possibly because his mathematical ethics were as rigid as those of Dedekind, who stated[11] the precept that "whatever can be proved in mathematics should not be believed without proof."

The materials for a proof were available to anyone after the late sixteenth century. Euclid[12] had taken the most difficult step, when he proved that if a prime divides neither of two integers, it does not divide their product. This also is obvious in the same deceptive sense that the fundamental theorem is obvious. The snare is in assuming that "prime" and "irreducible" or "indecomposable" necessarily have identical connotations. Gauss saw that the fundamental theorem requires proof and can be proved. His acuity in this detail alone put

him in a class above his predecessors and contemporaries in arithmetic. He seems to have been quite irritated (*ibid.* p. 9) by their vagueness in this and other fundamental matters.

Next there is the extraordinarily prolific concept of congruence, with its equally fruitful notation suggesting valid analogies with algebraic equations. This immediately gave arithmetic a new direction. It later (1847) did the like for algebra, when Cauchy[13] "realized" imaginaries by referring their induced real algebra to congruences modulo i^2+1, a tactic which Kronecker adopted and applied to algebraic number fields in general.

The relation between congruences and norms, though not mentioned in the *Disquisitiones*, possibly because it is almost trivial for rational arithmetic, was exploited by Gauss in connection with complete residue systems for Gaussian integers, specifically in his discussion of biquadratic residues.[7] This led others at the opportune time to the definition of the norm of an ideal. In retrospect, the transition from real to complex residue systems may appear to have been simple, and the means for effecting it natural. It would be interesting to know how much thought it cost Gauss. Whatever the effort, Gauss usually found the straight road to the future.

The ideas of mappings and correspondences, so prominent in modern algebra, entered abstract algebra by way of algebra itself rather than through arithmetic. Certainly homomorphism is nowhere recognized in Gauss' published writings as an independent concept, although instances of it are fully developed in the *Disquisitiones* and elsewhere in the theory of congruences. Dedekind's uses of mapping in his theory[3] of algebraic numbers (1871) were probably inspired by the Galois theory of equations, which he was one of the first to master, rather than by his critical study of the arithmetical works of Gauss, Dirichlet, and Kummer.

If it is mandatory in a sketch like the present to exhume a more or less ancient precedent for every mathematical principle mentioned and now in common use, this may easily be done for homomorphism. Both congruence and homomorphism, as usable abstractions, were commonplaces to the Pythagoreans of the sixth century B. C. Those ingenious numerologists remarked that all numbers, and therefore all things, are contained in the decad. This grand principle of mapping the entire universe on a double handful of positive integers was taken over by Plato and bodily incorporated into his peculiar substitute for science. It ultimately matured in his ideal numbers—not to be mistaken for Kummer's, though there are reasons for suspecting that Kummer had heard of Plato's celestial arithmetic. And if it is desirable to push recognition yet further back, it is sufficient to recall that the

Pythagoreans most likely got their decad from the finger-counting of savages.

In another direction, an isolated result in the *Disquisitiones* (Art. 42) is the extremely useful theorem now frequently called the "Gauss Lemma"—not to be confused with Gauss' Lemma for quadratic residues. It is to the effect that if A, B are polynomials in the indeterminate x, each with leading coefficient unity and the remaining coefficients rational numbers but not all integral, the coefficients in the distributed product AB are not all rational integers. An immediate corollary of this is more frequently used. Gauss in his Diary[14] (Item 69) dates the theorem 23 July 1797. The extension to algebraic numbers as coefficients was not far to seek, and Kronecker made powerful use of it in his theory of integral functions.

Another application of the extended Gauss Lemma has become classic in the theory of algebraic number ideals. The basic theorem that, if the ideal α divides the ideal γ, then there is an ideal β such that $\gamma=\alpha\beta$, or, what is equivalent, that for any ideal δ there is an ideal δ such that $\delta\delta'$ is a principal ideal, cost Dedekind in the late 1860's some effort to prove. In his simplified presentation of parts of Dedekind's theory, Hurwitz[15] obtained (1894) a painless proof based on the extended Lemma.

On a more abstract level, Gauss understood and freely applied the concept of indeterminates. In fact he is usually, but not always by those who have not dipped into his works, regarded as the originator of the concept; certainly nobody preceded him in this. Indeterminates permeate the theory of quadratic forms in the *Disqisitiones*, also the unpublished general theory (1797-8) of congruences.[16] Thus the form $ax^2+2bxy+cy^2$, for example, is symbolized (Art. 153) as (a,b,c); and binary forms are called "functions of two indeterminates x,y." The meaning of the term is carefully explained.

A perhaps more subtle type of use for indeterminates appears in Gauss' second proof (1815) for the so-called fundamental theory of algebra.[17] The critical detail of a "purely analytical" proof occurrred to Gauss (Diary, 29 February 1812) in November, 1811, in connection apparently, with his long struggle to complete the proof. Indeterminates are used as such in a proof (Art. 5) of the algebraic independence of the elementary symmetric functions. Here also (Art. 4) is the useful device of ordering the terms of a polynomial in any number of indeterminates lexicographically. Possibly the inner significance of indeterminates was not generally appreciated till Kronecker, in his arithmetical and algebraic philosophy, elevated them almost to the status of a religion.

Binary quadratic forms were to contribute to the development of algebraic numbers in another and possibly more tangible respect. Dirichlet[18] observed (1840) that the Gaussian theory of binary quadratic forms is properly a topic in multiplicative arithmetic. The full significance of this observation was perceived only when Dedekind[3] (1871) established a correspondence between binary quadratic forms of a given discriminant and the ideals of a quadratic field having the same discriminant, the composition of forms corresponding to the multiplication of ideals. The connection between the Gaussian class number, the class numbers for the relevant domains, and unique factorization or the lack of it for the integers concerned, then became apparent, as did also the reason for the existence of theorems giving the number of representations of rational integers in certain quadratic forms as closed formulas in terms of the divisors of the integers represented. The root of the connection was concealed in the Gaussian theory of composition for binary quadratic forms, and the composition of classes, orders, and genera of such forms.

Also in that theory occurs one of the earliest non-trivial arithmetical instances of an Abelian group: composition is the "multiplication" of the group. In his own terminology, Gauss discussed some features of the structure of the group of classes. Naturally he did not communicate his truly profound discoveries in the technical language of groups; for groups had not been officially invented in 1801, however numerous may have been almost trivial occurrences of them in primitive art no less than in elementary mathematics from Euclid to Gauss himself.

A somewhat labored attempt[19] by Poincaré (1885) to extend the Gauss-Dirichlet-Dedekind theory of binary quadratic forms to binary forms of degree higher than the second was only moderately successful, the substitute in terms of ideals for the original rational problem being at least as difficult as the original problem itself. If Gauss (or anyone else) had succeeded in constructing a usable theory of composition for the forms in question, some degree of practical success might have been anticipated. But Gauss never tried, and the problem is still open in all its unnaturalness. Although it may be difficult to characterize an artificial problem, it is generally agreed that Gauss seldom attacked one.

The significance of laws of reciprocity in the early development of algebraic numbers will be noted in another connection. For the moment it is sufficient to recall that the *Disquisitiones* (Arts. 135-45) contains the first complete proof of the Euler-Legendre law of quadratic reciprocity. Gauss records his success in obtaining a proof, after many trials, in his Dairy (Item 2), under the date 8 April 1796. The entry

for 29 April 1796 announces the generalization of the law rediscovered and published by Jacobi in 1837.

The fundamental theorem of algebra, already mentioned, also received its first acceptable proof (1799) at the hands of Gauss in his doctoral dissertation.[20] Although the theorem in its classical form, as stated and proved by Gauss, is no longer as important in algebra as it was, being replaced in the abstract theory by a purely algebraic equivalent that is almost a truism, the arithmetic of algebraic numbers could hardly have begun without it. Of the four proofs which Gauss devised, the second[21] was to become the most suggestive, especially for Kronecker's development of modular systems. The proof refers the existence of a root to an algebraic identity. From this, and the existence of a real root for an equation of odd degree with real coefficients, the theorem follows, provided certain assumptions of continuity be admitted.

As Gauss himself remarked, he had been anticipated by Euler and de Foncenex in the ingenious algebraic device underlying the proof; so possibly the credit for its suggestiveness should be transferred to them. It is interesting to recall that Clifford[22] (1876) independently outlined a somewhat similar proof. The theorem itself, as Gauss understood it, however proved, is now assigned to analysis rather than to the mathematics of the discrete, and its use in algebra is avoided.

Other items of importance for the development of algebraic numbers are the proof (*Disquisitiones*, Art. 341) of the irreducibility of $(x^n-1)/(x-1)$, n prime, and the like for the equation whose roots are the primitive mth roots of unity, m any positive integer. The last was obtained in 1808 but was not published. Its proof presupposes a knowledge—in a special instance—of the Dedekind inversion formula in numerical functions. A similar instance occurs in Gauss' theory (to be noted later) of what are now called Galois fields, generated by the residue systems of double moduli.

The preceding selection of items is by no means exhaustive of those in Gauss' early work which acquired a new significance with the advent of algebraic integers. But they are probably sufficient to indicate that Gauss' thought was in the main current of progress in algebra during the first four decades of the nineteenth century, although he himself may have been unaware of the particular direction progress was to follow.

5. *Recognition of algebraic integers.* It is no disparagement of Gauss' epochal achievement in introducting algebraic integers into arithmetic, to state the historical fact that there was no shadow of an attempt at an autonomous, general arithmetic of an arbitrary alge-

braic number field until Dedekind,[3] in the late 1860's succeeded with his theory of ideals. Kummer took a gigantic stride beyond Gauss when (1845) he created an arithmetic for the algebraic numbers that appear "naturally" in connection with Fermat's last theorem and higher reciprocity laws. But his ideal numbers were devised specifically for the integers of cyclotomic domains; and although Dedekind[2] gratefully acknowledged his initial indebtedness to Kummer's theory as a source of inspiration, his own general theory is of a radically different nature. Gauss might have imagined (and repudiated) ideal numbers—from his start in higher congruences; ideals, on the contrary, with their solution of a finite arithmetical problem by an appeal to infinite classes of integers, seem to belong to a realm of concepts alien to his mode of thought.

Nevertheless, Gauss took the first and decisive step. To appreciate the revolutionary character of what Gauss did in the early development of algebraic numbers, it is necessary to remember the sparse but significant hints in the work of his predecessors, and to bear in mind that both they and he were interested in irrational algebraic numbers solely for the possibility of applications to rational arithmetic.

One of the earliest applications of irrational algebraic numbers to rational arithmetic was the attempt by Euler (in his *Algebra*, 1770) to prove Fermat's assertion that the only solution of $x^2+2=y^3$ in rational integers is $x=5, y=3$. Euler assumed that $(x+i\sqrt{2})(x-i\sqrt{2})=y^3$ implies that each of $x \pm i\sqrt{2}$ is the cube of a number of the form $u+iv\sqrt{2}$, u,v rational integers. The assumption can be justified since the $u+iv\sqrt{2}$ constitute a unique factorization domain. This, of course, was unknown to Euler; and the failure of a similar device for the equation $2x^2-5=y^3$ (which has the solution $x=4$, $y=3$) was inexplicable at the time.

Almost simultaneously with Euler, and independently of him, Lagrange[23] conceived "the idea of using irrational and even imaginary factors of formulas [forms] of the second degree to find the conditions which render these [forms] equal to squares or to any powers." He remarks that in 1768 he read an unpublished memoir on the subject before the [Berlin] Academy; a resumé was printed in 1769. Gauss was familiar with this work. The detail of significance here is that the norm of a general algebraic number of degree n is a form of degree n in n indeterminates admitting composition. Incidentally, Lagrange reduced the general number to its canonical form. Elsewhere[24] he showed (1770) that the form $x^2+ay^2+bz^2+ab\,w^2$ repeats under multiplication, thus generalizing Euler's theorem (1748) in which $a=b=1$. Gauss derived Euler's theorem from multiplication of norms of complex numbers.[25]

No trace of a conjecture that irrational algebraic numbers might be subject to some of the laws of rational arithmetic has been detected in the writings of any predecessor of Gauss. What might have suggested the possibility to him? As he never explicitly told anyone, the only recourse is to examine the occurrences of algebraic integers in his work, much of which he did not publish. The relevant material is in those parts of mathematics in which Gauss worked, and to which, during his lifetime, a rapidly developing arithmetic of algebraic numbers was applied.

Three prolific sources of the theory of algebraic integers were attempts to prove Fermat's last theorem ($x^n+y^n=z^n$, $xyz\neq 0$, $n>2$ is impossible in rational integers x,y,z,n); the laws of $r-ic$ reciprocity, $r>2$; and the complex multiplication of certain special elliptic functions. It is known that Gauss (1777-1855) proceeded at different times from the first two. The first engaged his attention[26] as early as 1808; the second about 1814, when, according to his own testimony,[27] he obtained a proof of the law of biquadratic reciprocity. It is possible though improbable that biquadratic residues induced him to consider (Gaussian) complex integers as early as March, 1797, when he discovered the double periodicity of the lemniscate functions, whose study he had begun in January of the same year.[28] The third main historical source, complex multiplication, is problematical, so far as Gauss is concerned, although Jacobi, as will be seen later, suspected that Gauss had been led to (complex) algebraic integers through his discovery (unknown to Jacobi) of complex multiplication for the lemniscate functions.

The fragments[26] on Fermat's last theorem are inconsiderable in themselves, especially when contrasted with Dirichlet's and Kummer's works, or even with some of Lamé's. in the same direction. But in their intimations of a boundless new province of arithmetic they were of prophetic significance. Probably they were composed not later[29] than 1808. In the second, Gauss gives the material for a proof that Euclid's algorithm for the greatest common divisor of two integers in the natural domain R is applicable in the quadratic domain D defined by an imaginary cube root of unity, and hence that there is unique factorization of integers into primes in D. The unicity is assumed in the first fragment, where the impossibility of Fermat's equation is proved for $n=3$ in both R and D. A similar proof is outlined for $n=5$. For $n=7$ the devices that succeeded for $n=3,5$ fail, and Gauss queries "whether it is to be hoped that a proof may be obtainable from the properties of norms ["Determinanten", not "determinants"] and units ["Einheitszahlen"]"

The cases $n=3$, $n=5$ of Fermat's last theorem thus inspired the earliest known extensions of the fundamental theorem of arithmetic to domains other than the rational. The relevant fragments have the appearance of memoranda not intended for publication. This may account for the omission of definitions of "integer", "prime", and "unit" in the domains considered. Or it may have seemed to Gauss that the definitions were obvious and their explicit statement superfluous. In any event mathematics in 1808 was less precise than it gradually became after 1900, when Hilbert's insistence on axiomatics began to percolate.

In the first commentary (1825) on biquadratic residues, Gauss states[31] that he began to systematize the theories of cubic and biquadratic residues in 1805. From his posthumously published notes on cubic residues, it appears that he first approached both theories from the side of rational numbers and binary quadratic forms.[32] The forms required were actually norms of complex integers in the relevant quadratic fields. By 1808, Gauss had recognized the (complex) integers in the field generated by a complex cube root of unity, and was thereby enabled to formulate[44]* (implicitly) the law of cubic reciprocity in its simplest form.[33]

An entry in his Diary for 23 December 1808, records that Gauss began the study of the diophantine equation

$$x^3+ny^3+n^2z^3-3nxyz=\pm 1$$

on that day. The left member of this equation is the norm of the number $x+\nu y+\nu^2 z$ in the pure cubic field C defined by $\nu^3=n$, n a rational integer. He proved that n is a cubic residue of each divisor of any rational integer represented in the norm. The similarity between this property of the norm and the like for numbers represented in binary quadratic forms, may account for Gauss' invariable designation of norms as determinants.

Solutions of the diophantine equation furnish units in C. Without indicating the significance—from the standpoint of algebraic integers—of his calculations, Gauss recorded the units for certain values of n, and even in some instances exhibited the fundamental unit.[34]

Here is the earliest recorded hint that the "natural" generalization of the theory of binary quadratic forms, to which Gauss devoted so much of his early activity, is not to binary forms of degree higher than the second, but to the composable forms of degree n in n indeterminates arising as norms of algebraic integers of degree n. The fragment on the units in a pure cubic field is a step in the progression

*This note will appear in the next number.—EDITOR.

from Lagrange to Dirichlet,[18] the latter of whom in 1842-46 developed the general theory of algebraic units, incidentally pointing out that his theory includes and generalizes the theory of the Pellian equation. As Gauss had not published his own start in the same direction, Dirichlet was unaware of it. He proceeded directly from the algebraic developments of Euler and Lagrange; and it cannot be said that he was anticipated in any essential manner by Gauss.

From the fragments described and a very few others of a roughly similar nature on norms, it is clear that as early as 1808 Gauss had a practical knowledge of a few of the simpler integral domains with class number unity. The complete absence of notes on the factorization of integers in domains whose class number exceeds unity, may indicate that Gauss had satisfied himself that the Euclidean algorithm is not universally applicable, or, what for him may have been equivalent, that the fundamental theorem of arithmetic fails, in some of the domains he considered.[35] Either by almost miraculous good fortune or by mathematical insight, he did not commit himself in 1808 to plausible but erroneous inferences from an assumed universality of the fundamental theorem, as Lamé[36] and Cauchy[36] were to do in 1847. If the conjecture (by a prominent algebraist who prefers to remain anonymous), that the number of unique factorization domains of algebraic numbers is finite, should be substantiated, it would seem that luck alone was not responsible for Gauss' avoidance of the ingeniously contrived pitfall that deceived his eminent contemporaries.[36]

It is not known when, or why, Gauss lost interest in Fermat's last theorem. There is no hint in anything he published himself that he had ever given the theorem a thought. The notes on the cases $n=3,5,7$ have been dated 1808. It is a singular coincidence that in January of that year, Gauss[37] rather needlessly singled out Fermat for special suspicion of having stated results, found inductively, for which he either had no proof or deluded himself into believing he had. The occasion was an abstract of a new proof of the law of quadratic reciprocity. In the *Disquisitiones Arithmeticæ* (Art. 365), Gauss had already paid his disrespects to Fermat as an arithmetician. By stating only the less important half of the truth concerning Fermat's conjecture (1640) regarding the primality of $2^{2^n}+1$, Gauss succeeded in producing something dangerously close to a whole falsehood. In any event he did Fermat, who valued his reputation for probity more highly than his skill as a mathematician, a serious injustice.

Eight years after Gauss had stopped, baffled, before his special case $n=7$ of Fermat's last theorem, he disclaimed any serious interest in the theorem, but admitted by implication that he was still attracted by algebraic integers. This appears from his reply[38] to a letter of 7

March 1816 from Olbers, in which Olbers informed him that one of the prize questions proposed for 1818 by the Paris Academy of Sciences was a proof of Fermat's last theorem. Olbers remarked that the question seemed to him to have been specially made "für Sie, lieber Gauss." Gauss answered two weeks later. In view of all the circumstances, his reply can scarcely be considered a model of perspicuity and disingenuousness. The italics are in the original.

"Göttingen, 1816 March 21. I am much obliged to you for your communications concerning the Paris prizes. I confess indeed that the Fermat theorem as an isolated proposition has little interest for me, since a multitude of such propositions, which one can neither prove nor refute, can be easily promulgated. Nevertheless I am induced by this to take up again several old ideas for a *great* extension of the higher arithmetic. Admittedly this theory belongs to those things in which one cannot foresee how far it will succeed in reaching distant, [uncertainly defined] ends. A lucky star must also prevail; and my situation and so many distracting cocupations indeed do not permit me *so* to cling to such meditations as in the fortunate years 1796-1798, when I composed the main items of my *Disquisitiones Arithmeticæ*. However, I am convinced that, should luck do more than I may expect, and prosper me in certain main steps in that theory, then also the Fermat theorem will thereby appear as only one of the least interesting corollaries."

(To be concluded in the March issue)

REFERENCES, NOTES

W = C. F. Gauss, Werke; Crelle = Journal für die reine und angewandte Mathematik; Liouville = Journal des Mathématiques pures et appliquées.

[1] Mathematische Annalen, 36, 1890, 473-534.

[2] Bulletin des Sciences mathématiques et astronomiques, sér. 2, 1, 1877, 92.

[3] Supplement XI von Dirichlets Vorlesungen über Zahlentheorie, 4. Auflage, (1894), 434-657. (The first version appeared in the second edition, 1871). See also the Anzeige der zweiten ... Zahlentheorie, Göttingische gelehrte Anzeigen, Jhg. 1871, 1481-94. Dedekind began his project of creating a general theory in 1856, the year after the death of Gauss.

[4] See Dickson, History of the theory of numbers, 2, 1920, 738.

[5] Göttingische gelehrte Anzeigen, 1831 April 23; W 2, 181-178.

[6] See references 7, 34.

[7] Theoria residuorum biquadraticorum, Commentatio secunda, 1831, W 2, 102-148; §§31-76 contains the arithmetic of (Gaussian) complex integers.

[8] Crelle, 35, 1847, 319-367.

[9] Liouville 12, 1847, 202. Reprint of Kummer's Dissertation, 1844.

[10] Disquisitiones Arithmeticæ, Præfatio, especially VII-VIII.

[11] Was sind und was sollen die Zahlen, 1888, 1.

[12] Elements, Book VII, Prop. 32.

[13] Exercises d'analyse, etc., 4, 1847, 87-110.

[14] Facsimile and transliteration with comments in W 10, Abt. 1, 483-572.

[15] Göttingen Nachrichten, 1894, 291-8.

[16] W, 2, 212-240, posthumous.

[17] Reference 21.

[18] G. Lejeune Dirichlet's Werke, 1, 619-23, 633-44.

[19] Bull. Soc. Math. de France, 12, 1885, 162-94.

[20] W, 3, 1.

[21] W, 3, 105.

[22] Mathematical Papers of W. K. C., 1882, 20.

[23] Oeuvres 2, 527-32; 7, 170-9.

[24] Oeuvres, 3, 189-201, especially 201, ($-B$, $-C$ for a,b).

[25] W 3, 383-4.

[26] W 2, 387-91.

[27] W 2, 516; letter of 30 May 1828 to Dirichlet.

[28] Diary, Item 51.

[29] According to Bachmann, in W 10, Abt. 2, 60, the fragments belong to the last third of the year 1808. This he infers from Schering's note, W 3, 398. See reference 34.

[30] Reference 26, 391.

[31] W 2, 67. The first commentary was transmitted to the Royal Society of Göttingen in April 1825, and was printed in 1828; the second, transmitted in April 1831, was printed in 1832.

[32] W 8, 5-19.

[33] The law was assembled (1900) by Fricke[32]) in the course of his duties as an editor, from the membra disjecta strewn through Gauss' fragmentary notes. See reference 44.

[34] W 8, 21-24.

[35] See references 26, 32-4.

[36] See references 55, 57.

[37] W 2, 152.

[38] W 10, erste Abt., 75-6.

The Teachers' Department

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Random Jottings From An Instructor's Notebook

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Introduction. After spending fifteen years at the front of a mathematics classroom, one is careful not to make the claim that he is teaching mathematics. Certainly some of the results shown by tests would indicate the contrary. Nevertheless, from time to time one cannot help but discover certain items which have been of help in the presentation of material. Some have been original discoveries in the sense that no one else was involved, others have come through contacts with various teachers in other fields. On occasion, in casual conversation, I have learned from a colleague some valuable procedure—a technique which perhaps the colleague thought was common knowledge. It is my thought that although some of these random jottings may be known to many teachers, to others they may suggest a new idea, and so possibly lead to improvement in teaching methods.

The Purpose of a Question. During my first year of college teaching I was fortunate in having the opportunity for occasional conferences with an experienced and able professor. I shall never forget an interview after this professor had visited my class. "Mr. Read", he asked, "after pointing out that if the discriminant of a quadratic equation was negative, the roots were imaginary, what was your purpose in asking whether the discriminant itself could ever be imaginary?" My reply was to the effect that I wished to emphasize that the author was limiting the coefficients to the field of real numbers, and hence the discriminant could not itself be imaginary. "Excellent purpose," he replied, "and just what *did* your question accomplish?" Actually, it accomplished nothing except to confuse the class. It may be well, before putting a question, to ask yourself carefully: Will this question accomplish the desired purpose? It may be well, also, to analyze the situation afterwards with a view to possible improvement.

Mathematical Terminology. From the point of view of the teacher, some terms in mathematics do not always help to clarify a concept. In dealing with the number system, for example, there is an unfortunate connotation in the use of such terms as *irrational*, *imaginary*, and *surd* (which the student confuses with absurd). As another example, we might consider the expression "simplest form" as frequently applied to expressions involving radicals. Although his algebra text may have stated that the radical $1/\sqrt{2}$, if expressed in "simplest form", would be written as $\sqrt{2}/2$, the student may find a problem in analytics or calculus in which the first form would be simpler from the point of view of computation. It is probably poor practice for the teacher to attempt to use terminology of his own, yet I have found it helpful to point out that "simplest form" is in a sense synonomous with "standard form", and while it is usually the form best adapted for computation, this may not always be the case. If we consider the expression the equivalent of *standard form*, the result is that different workers will produce results which are readily comparable without further manipulation. Again, beginners are often confused by the fact that *cancel* is used in two senses—one where we simplify a fraction such as $3a/6b$ by *cancelling* the common factor 3, the other where we solve the equation $3x-5=1-4x-5$ by *cancelling* the term -5 on each side. I knew of one teacher who insisted upon using the terms *cancel* and *annul*, but I fear his students were confused when they encountered an instructor who did not use this terminology. Nevertheless, it is doubtless well worthwhile pointing out clearly the distinction in meaning.

Answer Books. Some teachers refuse to permit students to use answer books; others prefer that all answers be available to students, and a common practice is to furnish answers to approximately half the problems in a text. When answers are to be used, one must be careful to avoid the practice of working backward from the answer (although there are occasional situations in actual practice where a knowledge of the correct answer may yield a clue to a method which will solve other problems). As the courses become more advanced, the forms in which an answer may be written become more varied. In fact, when dealing with dfferential equations, it may be more work to show that two answers are equivalent than it was to obtain either answer. Nor are answer books always consistent in the form in which the answers are written. I have found it very helpful to suggest to students that they may consider that they are working in an office, engaged in the task of checking calculations made by the senior man in the office. The results as written in the answer book may be considered those obtained by the senior man. I then tell them: "Your

task is to show the senior man is right or wrong. Usually he will be right—in such cases you should put your answer in the same form as he did. It does not matter that you think your form is preferable, you are the junior man, and your task at present is to check, not criticize, his methods". The practice in transforming and rearranging together with an analysis of the reasons for writing the answer in a particular form, may be as valuable as solving the problem.

Rigorous Definitions. The standard of mathematical rigor which can be expected of students varies with the level at which a course is taught. Nevertheless, it does not seem unreasonable to expect that definitions used in earlier courses should be valid in later courses—particularly if exceptions or qualifying phrases are within the grasp of the student. For example, is there any reason why a high school student should not be given a definition of a cylinder which will include the cases of oblique, parabolic, or elliptical cylinders, even though he may then restrict his work to right circular cylinders? Or, when introducing the definitions of negative and zero exponents, does the teacher leave the impression that the definitions hold for any base, or is it pointed out that the definitions are not valid if zero is the base? Again, if the base a is affected by the fractional exponent p/q, is there any attempt at definition when a is negative and q even? An interesting mathematical fallacy may be studied by assuming that negative one with exponent 1/3 is the same as negative one with exponent 2/6.

Division by Zero. Even though as a teacher I recognize that division by zero is impossible, do my students recognize this fact? It may be worth while to spend half an hour on this concept. Let us start with the meaning of 12 divided by 3. We may assume $12/3 = x$, where x satisfies the condition that $3x = 12$. There is a unique x; its value is 4. Next proceed to such a problem as 18/7. If we assume $18/7 = x$, we may determine a unique x, although in this case it is not an integer. After several such examples, emphasizing the relation that the quotient is the number which, when multiplied by the divisor, produces the dividend, introduce the problem 0/5. Following the treatment, we require x, where $5x = 0$. It will be seen that there is a unique solution, $x = 0$. After several examples of this, one may introduce the problem 7/0. The requirement that $0x = 7$ will readily be seen to be impossible. It may take some careful questioning to draw out the fact that 0/0 is indeterminate. Analysis as before shows that we require $0x = 0$. Most students, if asked for the value of 0/0, will give either one or zero, but will soon agree that 5, 7, or any other number will also satisfy. It may be pointed out that when the divisor

is zero, the problem is either impossible or has no unique solution—justifying our exclusion of the operation of division by zero. If a student has seen a discussion similar to that just outlined, he is not likely to make the statement that $a/0$ equals infinity, or as once happened in my class, that $a/0+b/0$ equals 2 times infinity. When the student is ready to study elementary limiting processes, a very careful distinction should be made between the two indicated operations: (1) $5/0$ and (2) the limit of $5/x$ as x approaches zero. The first is an impossible operation when dealing with two constants. The result of the second operation is a variable which increases without bound. Careless use of the symbol for infinity may obfuscate this important distinction.

Scales on the X- and Y-Axes. It must be continually emphasized to the beginning student that the units used in graphing must be clearly indicated on the axes. Many students will object, insisting that it is obvious that each division on the graph paper represents one unit. A very effective answer is to present papers from two students, neither having the scale indicated, but with different assumptions as to the value of a division of the graph paper. It may also be pointed out that it is sometimes advisable to use different scales on the two axes, as, for example, when solving a cubic equation graphically. However, there is at least one situation where a choice of different scales on the X- and Y-axes will introduce awkward complications—the case of rotation of the axes in analytic geometry. To illustrate: if we wish to rotate through an angle whose tangent is $3/4$, is this also an angle whose sine is $3/5$? If so, what scale is used in measuring the five units if the scales along the two axes are not alike? If the teacher has not studied this problem, valuable information will be obtained by attempting to simplify a second degree equation by rotation and translation, using different scales on the two axes. Plot the equation in each system, and verify the fact that points whose coordinates in the original system satisfy the equation also have coordinates in the final system which satisfy the corresponding equation.

Transposition Errors. Almost every bookkeeper knows that if his accounts fail to balance by a sum divisible by nine, the error is likely to be a transposition error, and if so, the transposition tends to be a transposition of adjacent digits. It is less widely known that if the sum is divisible by nine, the quotient is of value in determining the error. Representing the tens digit by t and the units by u, the numerical value of a transposition error involving these digits is

$$(10t+u-10u-t)=9(t-u).$$

In other words, if we divide the error by nine, the quotient is the difference between the transposed digits. Hence, if the error is 72, the quotient after dividing by 9 is 8. The error, if one of transposition, could have resulted only from transposing 0 and 8 or 1 and 9. An error of 90, although divisible by nine, cannot be an error arising from transposing adjacent digits; an error of 63 can arise only by transposing 0 and 7 or 1 and 8, or 2 and 9. Extension to such figures as \$540, \$2.70, and the like is obvious. The rule and its application are both simple, and often hasten the location of an error.

Parametric Equations. Three brief jottings help me point out to the student facts which are not always clear in the text. First, although there may be one substitution which produces very simple parametric equations, it must be remembered that there is no *unique* method of obtaining parametric equations for a given curve. With a small class, this is brought out in a striking manner by assigning a simple rectangular equation to the group, giving a parametric equation for y—different for every member—and asking that they determine the parametric equation for x. Second, many texts give several problems in elimination of the parameter, but fail to point out that while the process may be extremely complicated or completely impossible for the student, yet the curve may be relatively simple and easy to plot from the parametric equations. Third, it should be emphasized that the graph given by a set of parametric equations may be only a portion of the graph obtained by eliminating the parameter. A very useful illustrative example is the graph represented by $x = \sin^2 t$ and $y = \cos^2 t$, which is obviously restricted to the first quadrant, while the rectangular equation obtained by eliminating t represents a line unlimited in extent.

Correct Usage of Terms. Attention has been called to the need for rigorous definitions of mathematical terms. Closely related and of equal importance is emphasis upon correct usage of terms. For example, it is not at all uncommon to find a calculus text using the term *pressure* as the equivalent of *force*. The instructor should not be surprised if some good student raises this issue—any attempt to defend the usage may result in difficulty. In the field of statistics one may discover that a city which increases in population ten per cent each year is said to have a constant rate of change. In caculus a constant rate of change means addition of an equal amount each year. The correct usage in statistics should probably be constant *ratio* rather than constant *rate* of change. Doubtless other illustrations can be added, but the essential point to remember is that when a term has a well-defined meaning, we should be careful that our usage is in conformity with this definition.

Alternative Definitions or Conventions. In many cases authorities are not in agreement as to definitions or conventions, as, for example, in the labeling of the axes in three dimensional analytic geometry; the admissibility of negative values for the radius vector in polar coordinates; whether or not the term *integer* includes zero; the definition of an asymtote; the definition of the principal values of certain inverse trigonometric functions; the distinction between the prolate and curtate cycloid. Without doubt there are reasons for the particular definition or convention selected. If the instructor prefers other alternatives, he may select another text or suggest the minor changes needed. However, it would seem that many authors are remiss in not pointing out to readers the fact that alternatives may exist. Here the instructor can in a few moments point out this fact and perhaps save the student much time and trouble in the situation where he attempts to use another reference and is puzzled by a new convention. There seems to be little excuse, to give only one example, for letting the student believe that the particular definition given in his trigonometry text for the principal value of arc csc x is a unique one. This is especially true if, as may happen, the calculus text in use in the same school gives a different definition.

Is a Problem Possible? A problem may be deliberately made impossible, such as the familiar puzzle: A train has to travel two miles; if it takes two minutes to travel the first mile, how long can it use in traveling the second mile if the average speed is 60 m.p.h.? Much more serious is the problem thrown together rather hastily, resulting in an absurd answer. I recall a problem in progressions, taken from one text, in which one had to solve a quadratic equation to find the number of terms of an arithmetic progression. The roots were $5\frac{1}{2}$ and -23. In a recently printed text in spherical trigonometry one finds a vessel sailing across the equator in Longitude 78°26′ W.; the fact that this is about 100 miles inland does not help the student gain confidence in the authors' ability as navigators. A high school text asks the student to write a formula for the cost, c, of shipping a package weighing n pounds by parcel post, if the charges are 11 cents for the first pound and five cents for each additional pounds. With proper notations, the problem is not impossible, but at the high school level, it is essentially impossible to write a formula giving the cost for a package of $7\frac{1}{4}$ pounds. To summarize: problems should be possible at the level of the student, and the answers should be both possible and plausible.

Directed Lines in Trigonometry. The definitions of the trigonometric functions of the general angle involve the use of the abscissa and ordinate, both of which are directed lines. For example, with the angle

in standard position, terminal line OP, and PM the perpendicular upon the X-axis, the sine of the angle is MP/OP and not PM/OP nor MP/PO; likewise the tangent is MP/OM and not PM/OM. Nevertheless, this point is often completely overlooked. Students beginning analytics see no difference in the two statements $\tan \theta = QP/RQ$ and $\tan \theta = PQ/RQ$, since the concept of directed lines has not been emphasized. If they are attempting to express PQ in terms of the difference in ordinates, difficulty arises in the failure to emphasize direction. If the text treats the line values of the trigonometric functions, the directed segment is always employed. Why, then, should it not be considered more carefully in the treatment of the functions as ratios?

Scale in Trigonometric Graphs. If the general shape is all that is required, it is not essential that the scales on the two axes be identical in plotting trigonometric curves. If one desires the proper proportions, or correct representation of area, the angle must be measured in radians (or an awkward change of scale must be made on the vertical axis). Trigonometry texts frequently mention the use of radian measure, but do not emphasize the reasons for such choice. When a calculus class first studies integration of trigonometric functions, it will be an interesting experiment to ask the class to draw an accurate graph of one arch of the sine curve, and estimate the area by counting squares on the coördinate paper. In some cases, because of failure to select the proper scale, discrepancies will be not only very large, but also very puzzling to the student.

Problems Illustrating the Theory of Logarithms. Some texts give problems which require the obtaining of logarithms of composite numbers, using only certain given logarithms of primes. For example, we find: Given that the common logarithm of $2 = 0.3010$; $\log 3 = 0.4771$; $\log 17 = 1.2304$, find (1) $\log 6$, (2) $\log 9$, (3) $\log 51$, (4) $\log 17/2$. Such problems have merit if properly used, but invariably some student discovers that instead of applying the theory, it is much easier to look up the required logarithms in a table. A minor variation which seems to retain all the merits of the problem would be to have the given logarithms to some base other than 10. Even if they are given as natural logarithms, a table is less likely to be available; if the instructor will go to the slight trouble of giving these values to some base such as 8, the student will find it necessary to apply the theory to obtain an answer, rather than using a table.

Problem Department

Edited by
E. P. STARKE and N. A. COURT

This department solicits the proposal and solution of problems by its readers, whether subscribers or not. Problems leading to new results and opening new fields of interest are especially desired and, naturally, will be given preference over those to be found in ordinary textbooks. The contributor is asked to supply with his proposals any information that will assist the editors. It is desirable that manuscripts be typewritten with double spacing. Send all communications to EMORY P. STARKE, Rutgers University, New Brunswick, N. J.

SOLUTIONS

No. 510. Proposed by *N. A. Court*, University of Oklahoma.

If the radical center of four given spheres coincides with the circumcenter O of the tetrahedron formed by the centers of the given spheres, the tetrahedron formed by the four polar planes of O for the given spheres admits O as the center of one of the spheres tangent to the faces of this tetrahedron. Conversely.

Solution by *Paul D. Thomas*, U. S. Navy.

The radical center O of the four given spheres (A), (B), (C), (D) is the center of a sphere (R), real or imaginary, orthogonal to these spheres. Now the two spheres (A), (R) being orthogonal, the polar plane of the center O of (R) for the sphere (A) is also the polar plane of the center A of (A) for the sphere (R). This polar plane is therefore perpendicular to OA at a point A' such that

$$\text{(a)} \qquad OA \cdot OA' = p$$

where p is the square of the radius of the orthogonal sphere (R). Similarly for the points B, C, D. Now, by assumption, we have $OA = OB = OC = OD$, whence it follows from (a) that OA' is equal to the analogous distances OB', OC', OD' relative to the vertices B, C, D, which proves the proposition.

Conversely. If the polar planes, with respect to the given spheres, of the radical center O of these spheres form a tetrahedron admitting O as the center of a sphere tangent to its faces, we have $OA' = OB' = OC' = OD'$. Hence we have, by (a), $OA = OB = OC = OD$, i. e., the

point O is then the circumcenter of the tetrahedron determined by the centers of the four given spheres.

Note. The coincidence of the radical center of the four given spheres with the center of their central sphere implies that the radical plane of any two of the given spheres coincides with the mediator of their line of centers, i. e., the two spheres are equal. Thus: *If the radical center of four spheres coincides with the center of their central sphere, the four given spheres are equal.*

The converse proposition is obviously true.

Also solved by the *Proposer.*

No. 513. Proposed by *E. P. Starke*, Rutgers University.

Assuming that all the roots of the cubic $x^3+ax^2+bx+c=0$ are real, show that the difference between the greatest and the least roots is not less than $(a^2-3b)^{\frac{1}{2}}$ nor greater than $2(a^2-3b)^{\frac{1}{2}}/3^{\frac{1}{2}}$.

Solution by *J. Ernest Wilkins, Jr.*, Tuskegee Institute.

Let $x_1 \geqq x_2 \geqq x_3$ be the three real roots of the cubic equation. Then we have

$$\begin{aligned} a^2-3b &= (x_1+x_2+x_3)^2-3(x_1x_2+x_1x_3+x_2x_3) \\ &= x_1^2+x_2^2+x_3^2-x_1x_2-x_1x_3-x_2x_3 \\ &= (x_2-\tfrac{1}{2}x_1-\tfrac{1}{2}x_3)^2+3(x_1-x_3)^2/4 \geqq 0, \\ (a^2-3b) &\geqq 3(x_1-x_3)^2/4, \\ x_1-x_3 &\leqq 2(a^2-3b)^{\frac{1}{2}}/3^{\frac{1}{2}}. \end{aligned}$$

We also have

$$\tfrac{1}{2}(x_3-x_1) \leqq x_2-\tfrac{1}{2}x_1-\tfrac{1}{2}x_3 \leqq \tfrac{1}{2}(x_1-x_3).$$

Therefore,

$$\begin{aligned} a^2-3b &\leqq \tfrac{1}{4}(x_1-x_3)^2+3(x_1-x_3)^2/4 = (x_1-x_3)^2, \\ x_1-x_3 &\geqq (a^2-3b)^{\frac{1}{2}}. \end{aligned}$$

It is clear that the upper bound of x_1-x_3 will be attained whenever x_2 is the arithmetic mean of x_1 and x_3 and that the lower bound will be attained whenever $x_1=x_2$ or $x_2=x_3$.

EDITORIAL NOTE. When all roots are equal, then $a^2=3b$ and the stated difference is obviously zero. In other cases there is an easy geometrical interpretation. $(a^2-3b)^{\frac{1}{2}}$ is the length of the segment cut off by the curve $y=x^3+ax^2+bx+c$ on the tangent at either the maxi-

mum point M or the minimum point N. $2(a^2-3b)^{\frac{1}{2}}/3^{\frac{1}{2}}$ is the segment cut off by the curve on a horizontal line through the inflection I. Consideration of the slope of the curve makes it evident that the length of the segment cut off by the curve on any horizontal line between M and N lies between these extremes. Since the X-axis is such a horizontal line, the proposition follows. It should also be noted that a^2-3b is $\frac{1}{4}$ the discriminant of the derivative of x^3+ax^2+bx+c.

No. 514. Proposed by *D. L. MacKay*, Evander Childs High School, New York.

Construct a right triangle of given perimeter so that the foot of the altitude on the hypothenuse shall divide that hypothenuse in extreme and mean ratio.

Solution by *Frank Hawthorne*, Allegheny College.

Divide the line AB in mean and extreme ratio at C. Describe a semicircle with AB as diameter. Erect a perpendicular at C meeting the semicircle at D. Draw AD and BD. Divide the given perimeter into three segments proportional to AB, BD, and AD. Construct a triangle having these segments as sides.

EDITORIAL NOTE. Let AC be the larger of the two segments AC, CB, so that $AC^2=AB\cdot CB$. On the other hand we have $BD^2=AB\cdot CB$, whence $BD=AC$. Thus: *if the hypothenuse of a right triangle is divided by the foot of the altitude in mean and extreme ratio, the larger segment of the hypothenuse is equal to the shorter leg of the triangle, and conversely.*

Now the ratio $AC:AB$ is equal to a numerical constant, hence the same is true of $BD:AB$; i. e. all the triangles whose hypothenuses are divided by the feet of their altitudes in mean and extreme ratio, are similar. This property justifies the above construction.—N. A. C.

No. 517. Proposed by *V. Thébault*, San Sebastian, Spain.

Consider triangles with integral sides such that the perimeter and the area are measured by the same number. Find all such triangles if further (1) two sides are consecutive integers; the same if (2) the three sides form an arithmetic progression.

Solution by *Earl V. Greer*, Bethany-Peniel College, Bethany, Okla.

From the hypothesis,

$$\sqrt{\frac{P}{2}\left(\frac{P}{2}-a\right)\left(\frac{P}{2}-b\right)\left(\frac{P}{2}-c\right)}=p,$$

(I) or $$(p-2a)(p-2b)(p-2c)=16p,$$

where a, b, and c are the sides of the triangle, and p is its perimeter. If p is an odd integer, the left side of equation (1) is odd, while the right side is even. Therefore p is even.

Let the sides of the triangle be a, $a+r$, and $2k-r$, where k and r are integers. Then $p=2(a+k)$, and equation (1) becomes

$$k(k-r)(a-k+r)=4(a+r),$$

(II) or $$a=\frac{k(k^2-2kr+r^2+4)}{k^2-kr-4}.$$

If $r=1$, $$a=\frac{k^3-2k^2+5k}{k^2-k-4}=k-1+\frac{8k-4}{k^2-k-4}.$$

Now $k^2-k-4\leq 8k-4$ if and only if $0\leq k\leq 9$. Within this range, only when $k=3$ or 9, is a positive and integral. These values for k determine two triangles, one with sides 5, 12 and 13, and the other with sides 9, 10, and 17.

If $2k=a$, the sides of the triangle become $2k$, $2k+r$, and $2k-r$, and equation (2) becomes

$$2k=\frac{k(k^2-2kr+r^2+4)}{k^2-kr-4},$$

which, when simplified, becomes $(k-r)(k+r)=12$, the only positive and integral solution of which is $k=4$, $r=2$. This determines a third triangle with sides 6, 8, and 10.

Also solved by *J. Ernest Wilkins, Jr.*

PROPOSALS

No. 548. Proposed by *N. A. Court*, University of Oklahoma.

Find the locus of the center of a variable sphere of fixed size which moves so that its radical plane with a fixed sphere passes through a fixed straight line.

No. 549. Proposed by *M. S. Robertson*, Rutgers University.

Let the distance from the center of the circle of radius r to the centroid of the area of a circular sector of angle 2θ be denoted by $f(\theta)$. In an elementary way show that $f(\theta)$ satisfies the functional equation

$$f(\theta)=f(\tfrac{1}{2}\theta)\cos(\tfrac{1}{2}\theta)$$

and obtain without calculus the solution

$$f(\theta) = \tfrac{2}{3}\, r \prod_{n=1}^{\infty} \cos(\theta/2^n) = \tfrac{2}{3}\, r\, \frac{\sin\theta}{\theta}.$$

No. 550. Proposed anonymously.

"What if, in No. 497 (NATIONAL MATHEMATICS MAGAZINE, Nov. 1943, p. 88), N_{j+1} is taken as covers N_j?" (That is: N_0 being any real number, if N_{j+1} = covers N_j, show that the limit of N_j as $j \to \infty$ is a fixed number independent of N_0 and find an approximation of its value.)

No. 551. Proposed by *W. V. Parker*, Louisiana State University.

Two given straight lines, s and s', intersect in O. Take A, B on s, on the same side of O, and A', B' on s', on the same side of O, such that $OA = OA'$, $OB = OB'$. Draw $AP \perp BP$, and $A'P' \| AP$ and $B'P' \| BP$. Show that (area OPP')^2 = area $OAA' \cdot$ area OBB'.

No. 552. Proposed by *E. P. Starke*, Rutgers University.

If B is an end of the minor axis of an ellipse, determine P on the ellipse such that the chord BP has a maximum length. What is the value of the eccentricity when P is an end of a latus rectum?

No. 553. Proposed by *Julius S. Miller*, Dillard University, New Orleans.

A flexible endless chain of mass M pounds rests on the surface of a smooth sphere of radius R feet, the chain forming a ring of radius r feet. Find the tension in the chain.